STANDARD 7 — GEOMETRY AND SPATIAL SENSE

K-12 Overview

All students will develop spatial sense and an ability to use geometric properties and relationships to solve problems in mathematics and in everyday life.

Descriptive Statement

Spatial sense is an intuitive feel for shape and space. It involves the concepts of traditional geometry, including an ability to recognize, visualize, represent, and transform geometric shapes. It also involves other, less formal ways of looking at two- and three-dimensional space, such as paper-folding, transformations, tessellations, and projections. Geometry is all around us in art, nature, and the things we make. Students of geometry can apply their spatial sense and knowledge of the properties of shapes and space to the real world.

Meaning and Importance

Geometry is the study of spatial relationships. It is connected to every strand in the mathematics curriculum and to a multitude of situations in real life. Geometric figures and relationships have played an important role in society's sense of what is aesthetically pleasing. From the Greek discovery and architectural use of the golden ratio to M. C. Escher's use of tessellations to produce some of the world's most recognizable works of art, geometry and the visual arts have had strong connections. Well-constructed diagrams allow us to apply knowledge of geometry, geometric reasoning, and intuition to arithmetic and algebra problems. The use of a rectangular array to model the multiplication of two quantities, for instance, has long been known as an effective strategy to aid in the visualization of the operation of multiplication. Other mathematical concepts which run very deeply through modern mathematics and technology, such as symmetry, are most easily introduced in a geometric context. Whether one is designing an electronic circuit board, a building, a dress, an airport, a bookshelf, or a newspaper page, an understanding of geometric principles is required.

K-12 Development and Emphases

Traditionally, elementary school geometry instruction has focused on the categorization of shapes; at the secondary level, it has been taught as the prime example of a formal deductive system. While these perspectives of the content are important, they are also limiting. In order to develop spatial sense, students should be exposed to a broader range of geometric activities at all grade levels.

By virtue of living in a three-dimensional world, having dealt with space for five years, children enter school with a remarkable amount of intuitive geometric knowledge. The geometry curriculum should take advantage of this intuition while expanding and formalizing the students' knowledge. In early elementary school, a rich, qualitative, hands-on study of geometric objects helps young children develop spatial sense and a strong intuitive grasp of geometric properties and relationships. Eventually they develop a comfortable vocabulary of appropriate geometric terminology. In the middle school years, students should begin to use their knowledge in a more analytical manner to solve problems, make conjectures, and look for patterns and generalizations. Gradually they develop the ability to make inferences and logical deductions based on geometric relationships and to use spatial intuition to develop more generic mathematical problem-solving skills. In high school, the study of geometry expands to address coordinate, vector, and transformational viewpoints which utilize both inductive and deductive reasoning. Geometry instruction at the high school level should not be limited to formal deductive proof and simple measurement activities, but should include the study of geometric transformations, analytic geometry, topology, and connections of geometry with algebra and other areas of mathematics.

At all grade levels, the study of geometry should make abundant use of experiences that require active student involvement. Constructing models, folding paper cutouts, using mirrors, pattern blocks, geoboards, and tangrams, and creating geometric computer graphics all provide opportunities for students to learn by doing, to reflect upon their actions, and to communicate their observations and conclusions. These activities and others of the same type should be used to achieve the goals in the seven specific areas of study that constitute this standard and which are described below.

In their study of **spatial relationships**, young students should make regular use of concrete materials in hands-on activities designed to develop their understanding of objects in space. The early focus should be the description of the location and orientation of objects in relation to other objects. Additionally, students can begin an exploration of symmetry, congruence, and similarity. Older students should study the two-dimensional representations of three-dimensional objects by sketching shadows, projections, and perspectives.

In the study of **properties of geometric figures**, students deal explicitly with the identification and classification of standard geometric objects by the number of edges and vertices, the number and shapes of the faces, the acuteness of the angles, and so on. Cut-and-paste constructions of paper models, combining shapes to form new shapes and decomposing complex shapes into simpler ones are useful exercises to aid in exploring shapes and their properties. As their studies continue, older students should be able to understand and perform classic constructions with straight edges and compasses as well as with appropriate computer software. Formulating good mathematical definitions for geometric shapes should eventually lead to the ability to make hypotheses concerning relationships and to use deductive arguments to show that the relationships exist.

The standard **geometric transformations** include translation, rotation, reflection, and scaling. They are central to the study of geometry and its applications in that these movements offer the most natural approach to understanding congruence, similarity, symmetry, and other geometric relationships. Younger children should have a great deal of experience with *flips*, *slides*, and *turns* of concrete objects, figures made on geoboards, and drawn figures. Older students should be able to use more formal terminology and procedures for determining the results of the standard transformations. An added benefit of experience gained with simple and composite transformations is the mathematical connection that older students can make to functions and function composition.

Coordinate geometry provides an important connection between geometry and algebra. Students can work informally with coordinates in the primary grades by finding locations in the room, and by studying simple maps of the school and neighborhood. In later elementary grades, they can learn to plot figures on a coordinate plane, and still later, study the effects of various transformations on the coordinates of the points of two- and three-dimensional figures. High-school students should be able to represent geometric transformations algebraically and interpret algebraic equations geometrically.

Measurement and geometry are interrelated, and an understanding of the **geometry of measurement** is necessary for the understanding of measurement. In elementary school, students should learn the meaning of such geometric measures as length, area, volume and angle measure and should be actively involved in the measurement of those attributes for all kinds of two- and three-dimensional objects, not simply the standard ones. Throughout school, they should use measurement activities to reinforce their understanding of geometric properties. All students should use these experiences to help them understand such principles as the quadratic change in area and cubic change in volume that occurs with a linear change of scale. Trigonometry and its use in making indirect measurements provides students with another view of the interrelationships between geometry and measurement.

Geometric modeling is a powerful problem-solving skill and should be used frequently by both teachers and students. A simple diagram, such as a pie-shaped graph, a force diagram in physics, or a dot-and-line illustration of a network, can illuminate the essence of a problem and allow geometric intuition to aid in the approach to a solution. Visualization skills and understanding of concepts will both improve as students are encouraged to make such models.

The relationship between geometry and **deductive reasoning** originated with the ancient Greek philosophers, and remains an important part of the study of geometry. A key ingredient of deductive reasoning is being able to recognize which statements have been justified and which have been assumed without proof. This is an ability which all students should develop in all areas, not just geometry, or even just mathematics! At first, deductive reasoning is informal, with students inferring new properties or relationships from those already established, without detailed explanations at every step. Later, deduction becomes more formal as students learn to use all permissible assumptions in a proof and as all statements are systematically justified from what has been assumed or proved before. The idea of deductive proof should not be confused with the specific two-column format of proof found in most geometry textbooks. The reason for studying deductive proof is to develop reasoning skills, not to write out arguments in a particular arrangement. Note that proof by mathematical induction is another deductive method that should not be neglected.

Much of the current thinking about the development of geometric thinking in students comes from the work of a pair of Dutch researchers, Pierre van Hiele and Dina van Hiele-Geldof. Their model of geometric thinking identifies five levels of development through which students pass when assisted by appropriate instruction.

- Visual recognition of shapes by their appearances as a whole (level 0)
- Analysis and description of shapes in terms of their properties (level 1)
- Higher "theoretical" levels involving informal deduction (level 2)
- Formal deduction involving axioms and theorems (level 3)
- Work with abstract geometric systems (level 4).

(Geddes & Fortunato, 1993)

Although the levels are not completely separate and the transitions are complex, the model is very useful for characterizing levels of students' thinking. Consistently, the research shows that appropriately targeted instruction is critical to children's movement through these levels. Stagnation at early levels is the frequent result of a geometry curriculum that never moves beyond identification of shapes and their properties. The discussion in this K-12 Overview draws on this van-Hiele model of geometric thinking.

IN SUMMARY, students of all ages should recognize and be aware of the presence of geometry in nature, in art, and in human-built structures. They should realize that geometry and geometric applications are all around them and, through study of those applications, come to better understand and appreciate the role of geometry in life. Carpenters use triangles for structural support, scientists use geometric models of molecules to provide clues to understanding their chemical and physical properties, and merchants use traffic-flow diagrams to plan the placement of their stock and special displays. These and many, many more examples should leave no doubt in students' minds as to the importance of the study of geometry.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Reference

Geddes, Dorothy, & Fortunato, Irene. "Geometry: Research and Classroom Activities," in D. T. Owens, Ed., *Research Ideas for the Classroom: Middle Grades Mathematics*. New York: Macmillan, 1993.

Standard 7 — Geometry and Spatial Sense — Grades K-2

Overview

Students can develop strong spatial sense from consistent experiences in classroom activities that use a variety of manipulatives and technology. The key components of spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

In kindergarten through second grade, the emphasis is on qualitative, not quantitative, properties of geometric objects. Students are at the visualization level of geometric thinking, where they perceive figures as "wholes". They recognize squares and rectangles, but perhaps not that squares are a special case of rectangles. To enrich and develop their geometric thinking, children at these grade levels need to explore geometry using a variety of physical objects, drawings, and computer tools. They work with solids, pattern blocks, templates, geoboards, and computer drawing tools to develop their understanding of geometric concepts and their spatial sense. They construct models and drawings to experiment with orientation, position, and scale, and to develop visualization skills. Students begin to develop a geometric vocabulary. A sample unit on geometry for the second-grade level can be found in Chapter 17 of this *Framework*.

In their study of **spatial relationships**, students focus on developing their understanding of objects in space. They discuss and describe the relative positions of objects using phrases like "in front of" and "on top of." They describe and draw three-dimensional objects in different relative locations. They compare and contrast shapes, describing the shapes of the faces and bases of three-dimensional figures. They discuss symmetry and look for examples of symmetry in their environment. They look for shapes that are the same size and shape (congruent) or the same shape but different sizes (similar). They use mirrors to explore symmetry.

In beginning their study of **properties of geometric figures**, students look for shapes in the environment, make models from sticks and clay or paper and glue, and draw shapes. They sort objects according to shape. They recognize, classify, sort, describe, and compare geometric shapes such as the sphere, cylinder, cone, rectangular solid, cube, square, circle, triangle, rectangle, hexagon, trapezoid, and rhombus. They describe the angle at which two edges meet in different polygons as being smaller than a right angle, a right angle, or larger than a right angle. They discuss points, lines, line segments, intersecting and non-intersecting lines, and midpoints of lines.

Students begin looking at **geometric transformations** by using concrete materials such as paper dolls to model slides (translations), flips (reflections), and turns (rotations). Students put shapes together to make new shapes and take shapes apart to form simpler shapes. Students work on spatial puzzles, often involving pattern blocks or tangrams. They look for plane shapes in complex drawings and explore tilings. They divide figures into equal fractional parts, for example, by folding along one, two, or three lines.

Coordinate geometry in grades K-2 involves describing the motion of an object. Students make maps of real, imaginary, or storybook journeys. They describe the location of an object on a grid or a point in a plane using numbers or letters. They give instructions to an imaginary “turtle” to crawl around the outline of a figure.

Students in these grades also begin to explore the **geometry of measurement**. In kindergarten, students discuss and describe quantitative properties of objects using phrases like “bigger” or “longer.” They order objects by length or weight. In first and second grade, they quantify properties of objects by counting and measuring. They determine the areas of figures by cutting them out of grid paper and counting the squares. They measure the perimeter of a polygon by adding the lengths of all of the sides.

Students begin to explore **geometric modeling** by constructing shapes from a variety of materials, including toothpicks and clay, paper and glue, or commercial materials. They use templates to draw designs, and record what they have constructed out of pattern blocks and tangrams. They fold, draw, and color shapes. They copy geoboard figures, and construct them both from memory and by following oral or written instructions. They may also use geometric models, such as the number line, for skip counting or repeated addition.

Geometry provides a rich context in which to begin to develop students’ **reasoning** skills. Students apply thinking skills in geometric tasks from identifying shapes to discovering properties of shapes, creating geometric patterns, and solving geometric puzzles and problems in a variety of ways. They create, describe, and extend geometric patterns. They use attribute blocks to focus on the similarities and differences of objects.

Geometry provides a unique opportunity to focus on the First Four Standards, especially Standard 2 which stresses the importance of making connections to other mathematical topics. For example, students often use their understanding of familiar shapes to help build an understanding of fractions. Teachers in grades K-2 need to plan classroom activities that involve several mathematical processes and relate geometry to other topics in mathematics. Geometry should not be taught in isolation; it should be a natural and integrated part of the entire curriculum.

Standard 7 — Geometry and Spatial Sense — K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Explore spatial relationships such as the direction, orientation, and perspectives of objects in space, their relative shapes and sizes, and the relations between objects and their shadows or projections.**

    - Blindfolded students are given real objects to touch and then, with the blindfolds removed, select the objects from a collection of visible objects.
    - Students work through the *Will a Dinosaurs Fit?* lesson that is described in the First Four Standards of this *Framework*. They discuss the size of the different dinosaurs and arrange them from smallest to largest.
    - Students predict what shape will result when a small piece is cut out of a folded piece of paper in different ways (along a diagonal, across a fold or a corner, or in the center) and the paper is then unfolded.
    - Students compare the sequence of objects seen from different points of view. For example, from the classroom window, the swings are to the left of the monkeybars, but the relationship is reversed if the objects are viewed from the blacktop facing the classroom.
    - Students predict and draw what the shadow of an object placed between a light and a screen will look like.

2. **Explore relationships among shapes, such as congruence, symmetry, similarity, and self-similarity.**

    - Students look for examples of congruent figures (same size and shape) in the environment.
    - Students explore symmetry by using mirrors with pattern blocks or by folding paper or by making inkblot designs. Students find the lines of symmetry in the letters of the alphabet and in numerals. They fold paper and cut out symmetric designs. They identify the symmetry in wallpaper or giftwrap designs.
    - Students use different size dolls and action figures as an introduction to the concept of similarity (same shape, different size).

3. **Explore properties of three- and two-dimensional shapes using concrete objects, drawings, and computer graphics.**

    - Students predict what shape they will see when they make various impressions of 3-dimensional objects in sand. For example, the top of a cylinder forms a circle, its side forms a rectangle.

- Students outline a triangle, a square, and a circle on the floor with string or tape. Then they walk around each figure, chanting a rhyme, such as "Triangle, triangle, triangle, 1, 2,3, I can walk around you as easy as can be," and counting the sides as they walk.
- Students work through the *Shapetown* lesson that is described in the First Four Standards of this *Framework*. They explore properties of two-dimensional shapes by applying the fundamental concepts of Venn diagrams.
- Some students use *Muppet™ Math* to work with Kermit's geometric paintings, while others use *Shape Up!* to compare everyday objects to geometric shapes.

4. Use properties of three- and two-dimensional shapes to identify, classify, and describe shapes.

- Students make shapes with their fingers and arms.
- Students listen to and look at the book *The Shapes Game* by Paul Rogers. Each page shows a different shape and many of the things in the world that have that shape. As each page is read, the children find other objects in the room that have the same shape.
- Students listen to and draw illustrations for the story *The Greedy Triangle* by Marilyn Burns.
- A good open-ended assessment for this critical indicator is to ask students to sort a collection of shapes into groups, explaining their reasoning. Some groups they might consider include "all right angles" or "four-siders." The teacher should encourage the students to invent appropriate group names and to use informal language to describe the properties, and should record the students' responses to look for progress over time.
- A more traditional, but still useful, assessment strategy is to ask students to sort pictures cut from magazines according to shape. This more focused task will generate information about the students' ability to recognize and differentiate among shapes.
- Students make class books shaped like a triangle, a rectangle, a square, and a circle. They fill each book with pictures of objects that have the shape of the book.
- Students turn a geometric shape into a picture. For example, a triangle might become a tower, a clown face, or the roof of a house.

5. Investigate and predict the results of combining, subdividing, and changing shapes.

- Students use tangram pieces to construct triangles, rectangles, squares, and other shapes.
- Students investigate which pattern block shapes can be formed from the equilateral triangles, recording their results in pictures and on a chart.
- Students work in groups to decide how to divide a rectangular candy bar among four people. The students then compare the various ways that each group solved the problem.

6. Use tessellations to explore properties of geometric shapes and their relationships to the concepts of area and perimeter.

- Students use Unifix cubes or pattern blocks to create colorful designs. They then discuss how many blocks they used (area) and the distance around their design (perimeter). They also discuss why these polygon shapes fit together like a puzzle.

- Students use different shapes to make quilt patterns.
- During free play time, students use pattern blocks to make different space-filling designs. They record any patterns that they especially like, using templates or drawing around the blocks.

7. Explore geometric transformations such as rotations (turns), reflections (flips), and translations (slides).

- Students look at the world around them for examples of changes in position that do not change size or shape. For example, a child going down a slide illustrates a slide, a merry-go-round or hands on a clock illustrate a turn, and a mirror illustrates a flip.
- Students look through and discuss the no-text book *Changes, Changes* by Pat Hutchins. In it, a man and a woman use the same set of building blocks to transform a house into a fire engine, then a boat, a truck, and back to a house. The students tell the story and then draw pictures to show how the blocks changed from one object to another.
- Students investigate the shapes that they can see when they place a mirror on a square pattern block.

8. Develop the concepts of coordinates and paths, using maps, tables, and grids.

- Students use maps of their community to find various ways to get from school to the park. They use letters and numbers to describe the location of the school and that of the park.
- Students create a map based on the familiar story of *The Little Gingerbread Man*, showing where each of the people in the story lives.
- Students describe how to get from one point in the school to another and try to follow each others' directions.

9. Understand the variety of ways in which geometric shapes and objects can be measured.

- In connection with a unit on dinosaurs in science, students discuss the different ways in which the size of dinosaurs can be described. They decide to measure the size of a dinosaur's footprint in two ways: by using string to measure the distance around it and by using base ten blocks to measure the space inside it.
- Pairs of students investigate the many different designs that they can make using unit squares and 1/2-unit right triangles. They record their results on dot paper.

10. Investigate the occurrence of geometry in nature, art, and other areas.

- Students take a "geometry walk" through their school or their neighborhood, looking for examples of specific shapes and concepts.
- Students create geometric patterns using potato prints.
- Students decorate their classroom for the winter holidays using geometric shapes.
- Students examine and discuss geometric patterns found in works of art.

References

Burns, Marilyn. *The Greedy Triangle*. New York: Scholastic, Inc., 1994.

Hutchins, Pat. *Changes, Changes*. New York: MacMillan, 1987.

Rogers, Paul. *The Shapes Game*. New York: Henry Holt and Company, 1989.

Software

Muppet™ Math. Jim Henson Productions. Sunburst Communications.

Shape Up! Sunburst Communications.

General Reference

Burton, G. and T. Coburn. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Kindergarten Book*. Reston, VA: National Council of Teachers of Mathematics, 1991.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 7 — Geometry and Spatial Sense — Grades 3-4

Overview

Students can develop strong spatial sense from consistent experiences in classroom activities that use a wide variety of manipulatives and technology. The key components of spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

In third and fourth grade, students are beginning to move beyond recognizing whole shapes to analyzing the relevant properties of a shape. They continue to use their own observations about shapes and the relations among these shapes in the physical world to build understanding of geometric concepts. Thus, using manipulative materials to develop geometric concepts and spatial sense remains important at these grade levels. Exploring concepts in a number of different contexts helps students to generalize. Students are extending their understanding of cause and effect and their ability to make conjectures. They are particularly interested in *Why?* Questions such as *Why are most rooms shaped like rectangles?* offer interesting points of departure for studying geometric concepts. Connections among geometry, spatial sense, other areas of mathematics, and other subject areas provide many opportunities for students to see how mathematics fits into their lives.

With respect to **spatial relationships**, students in these grade levels continue to examine direction, orientation, and perspectives of objects in space. They are aware of the relative positions of objects; you might ask *Which walls are opposite each other? What is between the ceiling and the floor?* Students also expand their understanding of congruence, similarity, and symmetry. They can identify congruent shapes, draw and identify a line of symmetry, and describe the symmetries found in nature. They search for examples in nature where each part of an item looks like a miniature version of the whole (self-similar).

Students are extending their understanding of **properties of geometric figures**. Now they are ready to discuss these more carefully and to begin relating different figures to each other. By experimenting with concrete materials, drawings, and computers, they are able to discover properties of shapes and to make generalizations like *all squares have four equal sides.* They use the language of properties to describe shapes and to explain solutions for geometric problems, but they are not yet able to deduce new properties from old ones or consider which properties are necessary and sufficient for defining a shape. They recognize the concepts of point, line, line segment, ray, plane, intersecting lines, radius, diameter, inside, outside, and on a figure. They extend the shapes they can identify to include ellipses, pentagons, octagons, cubes, cylinders, cones, prisms, pyramids, and spheres.

Students continue to explore **geometric transformations**. Using concrete materials, pictures, and computer graphics, they explore the effects of transformations on shapes.

Using coordinate geometry students create and interpret maps, sometimes making use of information found in tables and charts. Some grids use only numbers at these grade levels, while others use a combination of letters and numbers.

The geometry of measurement begins to take on more significance in grades 3 and 4, as students focus more on the concepts of perimeter and area. Students learn different ways of finding the perimeter of an object: using string around the edge and then measuring the length of the string, using a measuring tape, measuring the length of each side and then adding the measures together, or using a trundle wheel. They also develop non-formula-based strategies for finding the area of a figure.

Geometric modeling allows students to approach topics visually. For example, geometric shapes allow students to build an understanding of fraction concepts as they cut the shapes into congruent pieces. They can use the problem solving skill of drawing geometric diagrams, such as a polygon with its diagonals, to find out how many matches are played in a round robin tournament. They continue to build three-dimensional models of shapes, to draw two- and three-dimensional shapes with increasing accuracy, and to use computers to help them analyze geometric properties.

Students' use of **reasoning** continues to provide opportunities to connect geometry to Standards 1 - 4, to other areas of mathematics, to other disciplines, and to the real world. Students explain how they have approached a particular problem, share results with each other, and justify their answers.

Students in third and fourth grade are still dealing with geometry in a qualitative way but are beginning to adopt more quantitative points of view. They are able to use their natural curiosity about the world to expand their understanding of geometric concepts and spatial sense.

Standard 7 — Geometry and Spatial Sense — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Explore spatial relationships such as the direction, orientation, and perspectives of objects in space, their relative shapes and sizes, and the relations between objects and their shadows or projections.**

    - Students compare the sizes of the many shapes found in the classroom, such as the heights of students or the areas of their hands.
    - The teacher holds up a shape or describes a shape. Students locate this shape hidden in a box or bag containing a number of shapes, without looking at the shapes.
    - Students explore what happens to the shadow of a square when it is held at various angles to a beam of light. They continue their investigation with other two- and three-dimensional figures.
    - At half-hour intervals, students measure the length of the shadow of a stick stuck vertically into the ground.
    - Students trace the faces of a solid on a transparency and then challenge each other to identify the solid. They check their guess by bringing the solid to the overhead projector and placing it on each face in turn.
    - Students read *Ellipse* by Mannis Charosh. This one-concept book illustrates ellipses in all of their possible orientations and describes a variety of experiments that the students can perform to better understand the role of perspective in geometry.
    - Students predict the positions of three students from different points of view (perspective). For example, from the front of the room, they might see Joe on the left, Rhonda in the middle, and Carly on the right. From the back of the room, the positions would be reversed. Students find a perspective from which Rhonda is on the left, Carly is in the middle and Joe is on the right.

2. **Explore relationships among shapes, such as congruence, symmetry, similarity, and self-similarity.**

    - Students make a collection of natural shapes, including a wide variety of three-dimensional shapes such as fruits and vegetables, shells, flowers, and leaves. They describe the symmetry found in these shapes.
    - Students find objects that exhibit self-similarity, i.e., that contain copies of a basic motif which is repeated at smaller sizes of the same shape. Examples of such objects are feathers,

the shape of a coastline, chambered nautilus shells, and plants which branch out such as cauliflower, broccoli, Queen Anne's lace, and ferns.

- Students look for examples of congruent figures (same size and shape) in the environment.
- Students use scale models of cars and airplanes to study similarity. They recognize that figures that have the same shape but different sizes are similar.

3. Explore properties of three- and two-dimensional shapes using concrete objects, drawings, and computer graphics.

- Students look for a "Shape of the Day" throughout the school day, recording the number of times that the shape is seen.

- Students look for lines in the classroom, identifying pairs of lines that are parallel, that intersect, or that are perpendicular.

- Students use the computer language Logo to describe the path made by a turtle as it goes around different geometric shapes.

4. Use properties of three- and two-dimensional shapes to identify, classify, and describe shapes.

- Students make a chart or bar graph showing how many squares, rectangles, triangles, etc., they find in their classroom.

- Students "walk" a shape and have other students guess the shape.

- Students classify shapes according to whether they contain right angles only, all angles smaller than a right angle, or at least one angle larger than a right angle.

- One student thinks of a shape. The others ask questions about its properties, trying to guess it. For example, *Does it have a right angle?*

5. Investigate and predict the results of combining, subdividing, and changing shapes.

- Students investigate the shapes found in their lunches and then discuss how the shapes change as they nibble away. For example: *Can you change a four-sided sandwich into a triangle?*

- Students investigate how to use four triangles from the pattern blocks to make a large triangle, a four-sided figure, and a six-sided figure.

- Students combine tangram pieces to create a variety of shapes.

6. Use tessellations to explore properties of geometric shapes and their relationships to the concepts of area and perimeter.

- Students use square, triangular, and hexagonal grid paper to create colorful designs. They discuss why these polygon shapes fit together like a puzzle.

- Students use Unifix cubes or pattern blocks to create designs. They then discuss how many blocks they used (area) and the distance around their design (perimeter).

- Students work through the *Tiling a Floor* lesson that is described in the First Four Standards of this *Framework*. They discover that the shapes which can be used for tiling must be able to fit around a point without leaving spaces and without overlapping.

7. **Explore geometric transformations such as rotations (turns), reflections (flips), and translations (slides).**

   - Students use stuffed animals or two-sided paperdolls to show movements in the plane: slides, flips, and turns. They discuss how all slides (or flips or turns) are alike.
   - Students create borders from a single simple design element which is repeated using slides, flips, and turns.
   - Students study and describe the use of transformations in Pennsylvania Dutch hex signs, and then they design their own.
   - Students discuss transformations found in nature, such as the symmetry in the wings of a butterfly (a flip), the way a honeycomb is formed (slides of hexagons), or the petals of a flower (turns).
   - Students create quilt designs by using geometric transformations to repeat a basic pattern.

8. **Develop the concepts of coordinates and paths, using maps, tables, and grids.**

   - Students create Logo procedures for drawing rectangles or other geometric figures.
   - A good interdisciplinary assessment in both reading and mathematics is to have students draw maps for stories they have read, using coordinates to identify the locations of critical events or objects in the story.
   - Students find the lengths of paths on a grid, such as the distance from Susan's house to school.

9. **Understand the variety of ways in which geometric shapes and objects can be measured.**

   - Students discuss how to describe the size of a truck. Some suggestions include the length of the truck, its height (very important to know when it passes under another road), its cargo capacity (volume), or its weight (important for assessing taxes).
   - Each pair of students is given a pattern to cut out of oaktag and fold up into a three-dimensional shape. They are asked to measure the shape in as many ways as they can. They report their findings to the class.

10. **Investigate the occurrence of geometry in nature, art, and other areas.**

   - Students investigate the natural shapes that are produced by the processes of growth and physical change. They identify some of the simple basic shapes that occur over and over again in more complex structures. Students bring examples to class and describe the process in writing. Some interesting examples are honeycombs, pinecones, and seashells.
   - Students make a bulletin board display of "Shapes in the World Around Us."
   - Students read the beautifully illustrated book *Listen to a Shape* by Marcia Brown. The color photographs in the book move from the occurrence in nature of simple shapes to more complex ones. Children can be asked to describe and draw their favorite shapes in nature as a follow-up.
   - Students read *Shapes* by Phillip Yenawine. This carefully selected collection of works from

the Museum of Modern Art is analyzed to show how shapes contribute to the images on the canvas. An interesting open-ended assessment activity would be to ask the students to create their own works of art, combining the geometric shapes they know to make similar striking images.

References

Brown, Marcia. *Listen to a Shape*. New York: Franklin Watts, 1979.

Charosh, Mannis. *Ellipse*. New York: Thomas Y. Crowell, 1971.

Yenawine, Phillip. *Shapes*. New York: Delacourte Press, 1991.

Software

Logo. Many versions of Logo are commercially available.

General reference

Burton, G. et al. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Third-Grade Book*. Reston, VA: National Council of Teachers of Mathematics, 1992.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 7 — Geometry and Spatial Sense — Grades 5-6

Overview

Students can develop strong spatial sense from consistent experiences in classroom activities that use a wide variety of manipulatives and technology. The key components of spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

Informal geometry and spatial visualization are vital aspects of a mathematics program for grades 5 and 6. Middle school students experience the fun and challenge of learning geometry through creating plans, building models, drawing, sorting and classifying objects, and discovering, visualizing, and representing concepts and geometric properties. Students develop the understanding needed to function in a three-dimensional world through explorations and investigations organized around physical models.

Studying geometry also provides opportunities for divergent thinking and creative problem solving while developing students' logical thinking abilities. Geometric concepts and representations can help students better understand number concepts while being particularly well-suited for addressing the First Four Standards: problem solving, reasoning, making connections, and communicating mathematics.

Students' experiences in learning geometry should help them perceive geometry as having a dynamically important role in their environment and not merely as the learning of vocabulary, memorizing definitions and formulas, and stating properties of shapes. Students, working in groups or independently, should explore and investigate problems in two and three dimensions, make and test conjectures, construct and use models, drawings, and computer technology, develop spatial sense, use inductive and deductive reasoning, and then communicate their results with confidence and conviction. They should be challenged to find alternative approaches and solutions.

In their study of **spatial relationships**, students in grades 5 and 6 further develop their understanding of projections (e.g., top, front, and side views), perspectives (e.g., drawings made on isometric dot paper), and maps. They also consolidate their understanding of the concepts of symmetry (both line and rotational), congruence, and similarity.

Students expand their understanding of **properties of geometric figures** by using models to develop the concepts needed to make abstractions and generalizations. They focus on the properties of lines and planes as well as on those of plane and solid geometric figures. Students at this age begin to classify geometric figures according to common properties and develop informal definitions.

Still using models, drawings, and computer graphics, students expand their understanding of **geometric transformation**, including translations (slides), reflections (flips), rotations (turns), and dilations (stretchers/shrinkers). At these grade levels, the connections between transformations and congruence, similarity, and symmetry are explored. Students also begin to use **coordinate geometry** to show how figures change orientation but not shape under transformations. For these investigations they use all four quadrants of the coordinate plane (positive and negative numbers).

Students develop greater understanding of the **geometry of measurement** as they develop strategies for finding perimeters, areas (of rectangles and triangles), volumes, surface areas, and angle measures. The emphasis at this level should be on looking for different ways to find an answer, not simply on using formulas. Students use models for many problems, look for patterns in their answers, and form conjectures about general methods that might be appropriate for certain types of problems. Students apply what they are learning about areas to help them develop an understanding of the Pythagorean Theorem.

Students continue to use **geometric modeling** to help them solve a variety of problems. They explore patterns of geometric change as well as those involving number patterns. They use geometric representations to assist them in solving problems in discrete mathematics. They use concrete materials, drawings, and computers to help them visualize geometric patterns.

Students in these grade levels are beginning to develop more sophisticated **reasoning** skills. In studying geometry, they have many opportunities to make conjectures based on data they have collected and patterns they have observed. This inductive reasoning can then be related to what they already know; students should be encouraged to explain their thinking and justify their responses.

Throughout fifth and sixth grade, students use concrete materials, drawings, and computer graphics to increase the number of geometric concepts with which they are familiar and to explore how these concepts can be used in geometric reasoning. Students' natural curiosity about the world provides ample opportunities for linking mathematics with other subjects. The continued experience with two- and three-dimensional figures provided at these grade levels helps students build the firm foundation needed for the more formal geometry of the secondary school.

Standard 7 — Geometry and Spatial Sense — Grades 5-6

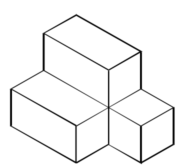
Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

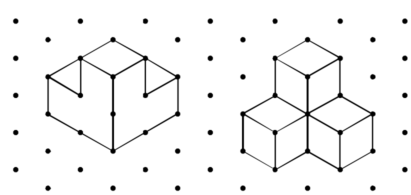
11. Relate two-dimensional and three-dimensional geometry using shadows, perspectives, projections, and maps.

- Students use centimeter cubes to construct a building such as the one pictured below. They then represent their building by drawing the base and telling how many cubes are stacked in each portion of building.

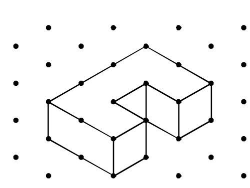


2	2	1
1	1	

- Students put three or four cubes together to make a solid and draw two different projective views of the solid on triangle dot paper, such as those shown below.



- Students copy pictures of solids drawn on triangle dot paper such as the one below, build the solids, and find their volumes.



- Students use circles and rectangles to make 3-dimensional models of cylinders, cones, prisms, and other solids.
- Students predict and sketch the shapes of the faces of a pyramid, or, given a flat design for a

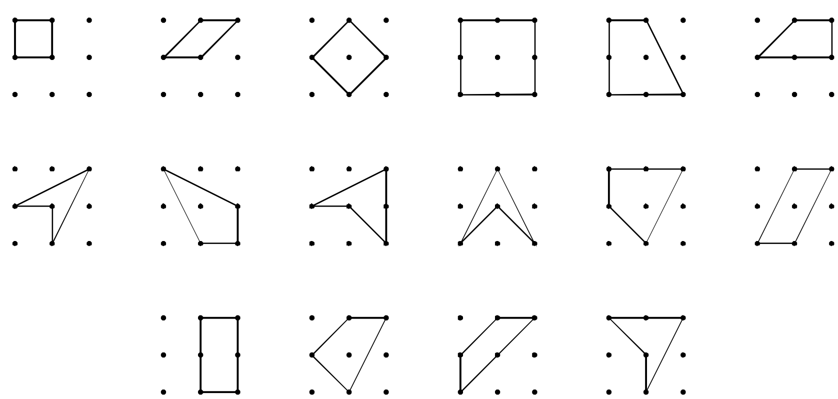
box, predict what it will look like when put together.

12. Understand and apply the concepts of symmetry, similarity, and congruence.

- Students compare different Logo procedures for drawing similar rectangles.
- Students look for examples of congruent figures (same size and shape) in the environment.
- Students explore symmetry by looking at the designs formed by placing a mirror on a pattern block design somewhere other than the line of symmetry, or by folding paper more than one time. They identify the symmetry in wallpaper or giftwrap designs. They also identify the rotational symmetry found in a pinwheel (e.g., 90°). (The figure matches itself by turning rather than by flipping or folding.)
- Students build scale models to investigate similarity. They recognize that figures which have the same shape but different sizes are similar.

13. Identify, describe, compare, and classify plane and solid geometric figures.

- Students are given a sheet of 3 x 3 dot paper grids. They find and draw as many noncongruent quadrilaterals as they can, using a different set of nine dots for each figure; altogether sixteen different quadrilaterals (pictured below) can be formed.



  A nice open-ended approach to assessment of their understanding and comfort with properties of geometric figures is to ask them to sort these quadrilaterals in different ways, including concave vs. convex, by angle sizes, by area, by symmetry, and so on. See how many ways they can devise.

- Students trace a figure onto several transparencies; figures such as squares, rectangles, parallelograms, rhombuses, trapezoids, kites, and arrowheads can be used. Then they draw the lines of symmetry for the figure. They rotate, translate, and flip the transparencies and compare them to an original transparency to investigate such properties as: the number of congruent sides in the figure, the number of parallel sides in the figure, whether the diagonals are congruent, whether the diagonals bisect each other, whether the diagonals are perpendicular, and whether the figure has half-turn symmetry (180°). They write about their findings and explain their reasoning.
- Students use Logo to investigate the sum of the measures of the exterior angles of any

polygon (360°) and the angle measure of each exterior angle of a regular polygon.

- Students select straws cut to five different lengths (for example, from one inch to five inches) and form as many different triangles as they can, recording the results. They sort the triangles into groups with 0, 2, or 3 equal sides and label the groups as scalene, isosceles, and equilateral triangles.

14. Understand the properties of lines and planes, including parallel and perpendicular lines and planes, and intersecting lines and planes and their angles of incidence.

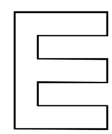
- Students use index cards with slits cut in them to build models of two planes that are parallel or two planes that intersect (in a line).
- Students use toothpicks to explore how two lines might be related to each other (parallel, intersecting, perpendicular, the same line).
- Students find examples of parallel lines and planes, perpendicular lines and planes, and intersecting lines and planes with different angles in their environment.

15. Explore the relationships among geometric transformations (translations, reflections, rotations, and dilations), tessellations (tilings), and congruence and similarity.

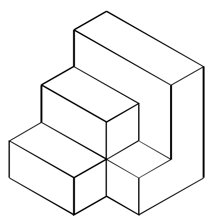
- Students read and examine *The World of M.C. Escher* or any other collection of M.C. Escher's work to find and describe the tessellations in them. Transformations of tessellating polygons are then performed by the students to make their own artwork.
- Students create a design on a geoboard, sketch their design, move the pattern to a new spot by using a specified transformation, and sketch the result.
- Students investigate wallpaper, fabric, and gift wrap designs. They create a template for a unit figure which they will use to create individual border designs for their classroom. Each student presents her/his design to the class, describing the transformations used to create the design.
- Working in small groups, students tile a portion of their desktop using oak-tag copies they have cut of a shape they have created by taping together two pattern blocks. Each group presents its results. The teacher then asks the students to compare the results of the different groups and identify examples of the different transformations used.
- Students investigate how transformations affect the size, shape, and orientation of geometric figures. A reflection or flip is a mirror image. A translation or slide moves a figure a specified distance and direction along a straight line. A rotation or turn is a turning motion of a specified amount and direction about a fixed point, the center. These transformations do not change the size and shape of the original figure. However, a dilation enlarges (stretches) or reduces (shrinks) a figure, producing a new figure with the same shape but a different size.
- Students use *Tesselmania!* software to manipulate and transform colorful shapes on the computer screen and create complex tessellations.
- Students continue to look for and report on transformations they find in the world around them.

16. **Develop, understand, and apply a variety of strategies for determining perimeter, area, surface area, angle measure, and volume.**

- Students are given a transparent square grid to place over a worksheet with triangles drawn on it. Using the grid to measure, they find the base, height, and area of each triangle, recording their findings in a table. They discuss patterns that they see, developing their own formula to find areas of triangles.
- Students find the perimeter of a figure by taping a string around it and then untaping and measuring the string. For something large, like the classroom, they might construct and use a trundle wheel.
- Students first estimate the perimeter (or area, volume, or surface area) of a classroom object, then measure it, determine its perimeter, and compare their answers to their estimates. Objects which might be used include books, desks, closets, doors, or windows.
- Students use various same-shape pattern blocks and arrange as many as are needed around a point to complete a circle. They discover the size of each angle since there are 360° in one circle. For example, if it takes six (green) triangles, then each angle must be 60° (360° ÷ 6).
- Students are given a sheet with rectilinear figures (only right angles) on it, such as the letter "E" at the right, and a transparent square centimeter grid that they can place over each of the figures. By counting the squares, they can find the area of each figure; by counting the number of units around it, they can determine its perimeter.



- Students use centimeter cubes to build a structure such as the one shown below and then count the cubes to find the volume of the structure.



- Students bring cereal boxes from home, cut them apart, and determine their surface areas.
- Students find the volumes of different backpacks by using familiar solids to approximate their shape. They compare their results and write about which backpack they think would be "best" and why.

17. **Understand and apply the Pythagorean Theorem.**

- Students construct squares on each side of a right triangle on a geoboard and find the area of each square. They repeat this process using several different triangles, recording their results in a table. Then they look for patterns in the table.
- Students measure the distance diagonally from first to third base on a baseball field and compare it to the distance run by a player who goes from first to second to third. They note that it is a shorter distance diagonally across the field than it is along the two sides. They

repeat this type of measuring activity for other squares and rectangles, noting their results in a table and discussing any patterns they see. They calculate the square of each of the three sides of each triangle, record their results in a table, and look for patterns.

18. Explore patterns produced by processes of geometric change, relating iteration, approximation, and fractals.

- Students use the reducing and/or enlarging feature on a copier to explore repeated reductions/enlargements by the same factor (iteration).
- Students investigate the natural shapes that are produced by growth. They look at how nature produces complex structures in which basic shapes occur over and over. For example, spider webs, honeycombs, pineapples, pinecones, nautilus shells, and snowflakes grow larger in a systematic way (iteration).

19. Investigate, explore, and describe the geometry in nature and real-world applications, using models, manipulatives, and appropriate technology.

- Students design a three-dimensional geometric sculpture. Some may want to find plans for making a geodesic dome and construct it out of gumdrops and toothpicks.
- Students work through the *Two-Toned Towers* lesson that is described in the First Four Standards of this *Framework*. They use models to determine how many different towers can be built using four blocks of two different colors.
- Groups of students working together design a doghouse to be built from a 4' x 8' sheet of plywood. They construct a scale model of their design from oaktag.
- Students use computer programs like *The Geometry PreSupposer* to explore the relationships of sides of polygons or properties of quadrilaterals.
- Assessments that make use of manipulatives and computer software allow students to demonstrate their knowledge and understanding of geometry. The results of performance tasks such as the following would be appropriate for a portfolio: *Make as many different sized squares as you can on a five-by-five geoboard. Create a tessellation pattern with pattern blocks or Tessellmania! software that uses slides, flips, and turns.*
- Students select a country or culture, research the use of specific geometric patterns in that culture, and make a report to the class.
- Specific manipulatives that may be helpful for geometry include pattern blocks, color tiles, linking cubes, centimeter cubes, tangrams, geoboards, links, and templates. Computer programs such as *Logo, Shape Up!, Elastic Lines, Building Perspective,* or *The Factory* may also be helpful.

References

Looher, J. L, Ed. *The World of M. C. Escher*. New York: Abradale Press, Harry N. Abrams, Inc., 1971, 1988.

Software

Building Perspective. Sunburst Communications.

Elastic Lines. Sunburst Communications.

Logo. Many versions of Logo are commercially available.

Shape Up! Sunburst Communications.

Tesselmania! Minnesota Educational Computing Consortium (MECC).

The Factory. Sunburst Communications.

The Geometry PreSupposer. Sunburst Communications.

General references

Diggins, Julia. *String, Straightedge, and Shadow: The Story of Geometry*. New York: Viking Press, 1965.

Geddes, D. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Geometry in the Middle Grades*. Reston, VA: National Council of Teachers of Mathematics, 1992.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 7 — Geometry and Spatial Sense — Grades 7-8

Overview

Students can develop strong spatial sense from consistent experiences in classroom activities that use a wide variety of manipulatives and technology. The key components of spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

Students in grades 7 and 8 learn geometry by: engaging in activities and spatial experiences organized around physical models, modeling, mapping, and measuring; discovering geometric relationships by using mathematical procedures such as drawing, sorting, classifying, transforming, and finding patterns; and solving geometric problems.

Building explicit linkages among mathematical topics is especially important with respect to geometry, since geometric concepts contribute to students' understanding of other topics in mathematics. For example, the number line provides a way of representing whole numbers, fractions, decimals, integers, lengths, and probability. Regions are used in developing understanding of multiplication, fraction concepts, area, and percent. The coordinate plane is used to relate geometry to algebra and functions. Similar triangles are used in connection with ratio and proportion.

Students continue to develop their understanding of **spatial relationships** by examining projections (viewing objects from different perspectives), shadows, perspectives, and maps. They apply the understanding developed in earlier grades to solve problems involving congruence, similarity, and symmetry.

Students begin to explore the logical interrelationships among previously-discovered **properties of geometric figures** at these grade levels. They extend their work with two-dimensional figures to include circles as well as special quadrilaterals. They continue to work with various polygons, lines, planes, and three-dimensional figures such as cubes, prisms, cylinders, cones, pyramids, and spheres.

The study of **geometric transformations** continues as well at these grade levels, becoming more closely linked to the study of algebraic concepts and **coordinate geometry** in all four quadrants. Students begin to represent transformations and/or their results symbolically. They also continue to analyze the relationships between figures and their transformations, considering congruence, similarity, and symmetry.

The geometry of measurement is extended to circles, cylinders, cones, and spheres in these grades. Students learn about π and use it in a variety of contexts. They explore different ways to find perimeters, circumferences, areas, volumes, surface areas, and angle measures. They also develop and apply the Pythagorean Theorem. The emphasis is on understanding the processes used and on recording the procedures in a formula; students should not simply be given a formula and be expected to use it.

Students continue to use **geometric modeling** to represent problem situations in different areas. Drawings of various types are particularly useful to students in understanding the context of problems. Number lines, coordinate planes, regions, and similar triangles help students to visualize numerical situations. Especially important are the patterns produced by change processes, including growth and decay.

Students further develop their **reasoning** skills by making conjectures as they explore relationships among various shapes and polygons. For example, as students learn about the midpoints of line segments, they can make guesses about the shapes produced by connecting midpoints of consecutive sides of quadrilaterals. By testing their hypotheses with drawings they make (by hand or using a computer), the students come to actually see the possibilities that can exist. The informal arguments that students develop at these grade levels are important precursors to the more formal study of geometry in high school.

The emphasis in grades 7 and 8 should be on investigating and using geometric ideas and relationships, not on memorizing definitions and formulas. A special feature of these grade levels is that students are preparing to take the New Jersey Early Warning Test (EWT). Many of the items in the Measurement and Geometry Cluster of the EWT will ask students to use those geometric ideas and relationships to solve problems – not simply to recall formulas. Indeed, the formulas needed for the problems are given to them on the Reference Sheet that accompanies the test. In their general classroom activity, as well as in preparation for the EWT, students should use a variety of concrete materials to model and analyze situations in two and three dimensions. They should use drawings that they make, either by hand or with the aid of a computer, to further examine geometric situations or to record what they have done. Geometry approached in this way can be fun and challenging for students.

Standard 7 — Geometry and Spatial Sense — Grades 7-8

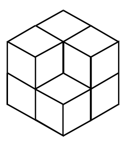
Indicators and Activities

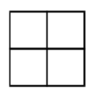
The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

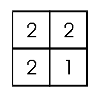
Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

11. Relate two-dimensional and three-dimensional geometry using shadows, perspectives, projections, and maps.

- Students build and draw solids made of cubes. They learn to build solids from drawings and to make their own drawings of solids. Among the drawings with which they should be familiar are the two-dimensional flat view from top, front, and side; the three-dimensional corner view; and the map view showing the base of the building with the number of cubes in each stack. For example, they can build the solid below; presented here are a three-dimensional corner view, a flat view, and a map view.







- Students predict what the intersections of a plane with a cylinder, cone, or sphere will be. Then they slice clay models to verify their predictions.
- Students use cubes made of clear plexiglas and partially filled with colored water to investigate cross sections of a plane with a cube. They try to tilt the cube so that the surface of the water forms various shapes, such as a square, a rectangle that is not a square, a trapezoid, a hexagon, and others.

12. Understand and apply the concepts of symmetry, similarity, and congruence.

- Students create three-dimensional symmetric designs using cubes, cylinders, pyramids, cones, and spheres.
- Students build scale models of the classroom, using similarity to help them determine the appropriate measures of the models.
- Students use compasses and straight-edges to construct congruent line segments and angles.
- Students work through the *Sketching Similarities* lesson that is described in the First Four Standards of this *Framework*. Students use a computer program and various similar figures to discover that corresponding angles have equal measures and corresponding sides have equal ratios.

13. Identify, describe, compare, and classify plane and solid geometric figures.

- Students use toothpicks to construct as many different types of triangles as possible, where each side of the triangles consists of between one and five toothpicks. They record their findings in a table, showing how many triangles are scalene, isosceles, equilateral, right, and obtuse. They also indicate which combinations of sides are impossible.

- Students sort collections of quadrilaterals according to the number of lines of symmetry that each has.

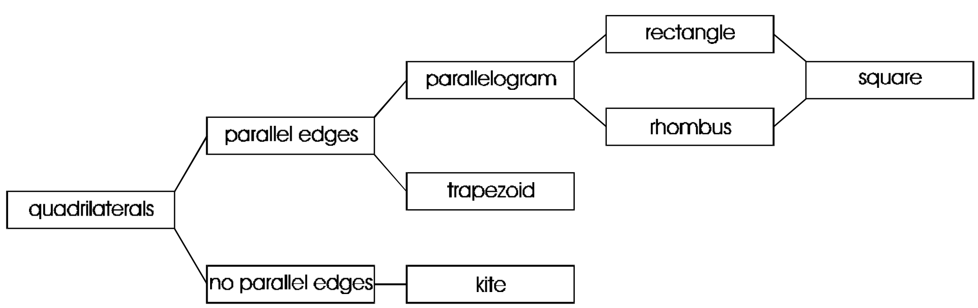
- Students play clue games designed to help them distinguish between necessary and sufficient conditions in describing a shape. For example: *If you want to challenge your friend to identify a square by giving a set of clues, which minimum set of clues would you select from the list below? Explain your selection. Is it possible to select a different minimum set of clues? Explain.*

— 4 right angles	— all angles are 90°
— 4 sides	— all sides of equal length
— all angles congruent	— opposite angles congruent
— opposite sides parallel	— simple closed curve

- Open-ended assessment items like those used on the Early Warning Test can always be used to provoke discussion and classroom activity. One of the sample items in the New Jersey Department of Education's *Mathematics Instructional Guide* (MG3) shows several figures and asks which of them can be put together to form a square. The developmental and extension activities provided there offer good suggestions for manipulative and transformation tasks.

- Students work through the *A Sure Thing !?* lesson that is described in the Introduction to this *Framework*. They investigate the relationship among the measures of the interior angles of a triangle by cutting out arbitrary triangles, tearing them into three pieces so that each corner is intact, and fitting the corners around a single point to make a straight angle.

- Students use diagrams to demonstrate the relationships among properties. For example, they might draw a diagram to show the logical relationship of ideas leading to the angle sum for a quadrilateral, or, as below, to clarify the relationship among different types of quadrilaterals.

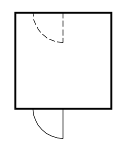


14. Understand the properties of lines and planes, including parallel and perpendicular lines and planes, and intersecting lines and planes and their angles of incidence.

- Students build a model of a cube, connect a midpoint of an edge with a midpoint of another edge, and then connect two other midpoints of edges to each other. They describe the relationships of the segments they have constructed. They change one of the line segments to another location and repeat the activity.
- Students identify congruent angles on a parallelogram grid, using their results to develop conjectures about alternate interior angles and corresponding angles of parallel lines and about opposite angles of a parallelogram.
- Working in groups, students review geometric vocabulary by sorting words written on index cards into groups and explaining their reasons for setting up the groups in the way that they did.

15. Explore the relationships among geometric transformations (translations, reflections, rotations, and dilations), tessellations (tilings), and congruence and similarity.

- Students use the “nibble” technique to create a shape which will tessellate the plane, that is, copies of this shape will fit together to cover a planar surface like a sheet of white oak-tag. Start with a square, cut off a “nibble” along the top or bottom edge of the square and translate the nibble vertically to the opposite edge of the square; the “nibble” will then be outside the boundary of the original square. Take a “nibble” from the right or left edge of the square and translate it horizontally to the opposite edge of the square. Trace this shape repeatedly onto a sheet of white oak-tag, by interlocking the pattern, and decorate the copies of the shape. Attempt this process several times until a pleasing shape is created.



- Students analyze the patterns found in Arabic designs such as tiled floors and walls in Spain, identifying figures that represent translations, reflections, and rotations. Then they generate their own tile designs using basic geometric shapes. They can create their own tile patterns using *Tesselmania!* software.
- Students apply transformations to figures drawn on coordinate grids, record the coordinates of the original figure and its image, and look for patterns. They express these patterns verbally and symbolically. For example, flipping a point across the x-axis changes the sign of the y-coordinate so that the point (x,y) moves to (x, y).
- Students practice doing geometric transformations mentally by using the *Geometric Golfer* or similar computer software. These programs present a series of puzzles in which there is an *object* shape and a *target* shape. The task is to use the fewest transformations possible to change the *object* shape so that it is congruent to the *target* shape. In the golf game, the object is a ball and the target is a hole.

16. **Develop, understand, and apply a variety of strategies for determining perimeter, area, surface area, angle measure, and volume.**

- Students use a paper fastener to connect two models of rays to form angles of different sizes. They estimate the correct position, then measure their guess with a protractor to see how close they were.
- Students are given a parallelogram-shaped piece of oak-tag and asked to cut it apart and arrange the parts so that it is easy to find its area. Their solutions are expressed verbally and symbolically. This same process is repeated for a trapezoid.
- Students bring cans from home, arrange them by estimated volume from smallest to largest, determine the actual volumes by measuring and computing, and compare these results to their estimates.
- Good conceptual assessment items designed to measure students' understanding of area frequently ask the students to find the area remaining in one figure after the area of another figure has been removed. One sample item from the New Jersey Department of Education's *Mathematics Instructional Guide*, for example, asks students to find the area of a circular path that surrounds a circular flower garden (MG1). Problems like this one are not only good practice for tests like the Early Warning Test but can also be used as informal assessments by teachers who listen carefully to their students' discussions about them.
- Students work through the *Rod Dogs* lesson that is described in the First Four Standards of this *Framework*. Students determine the effects of increasing the dimensions of an object on its surface area and volume.

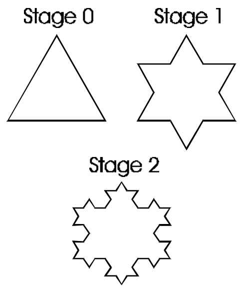
17. **Understand and apply the Pythagorean Theorem.**

- Students draw right triangles on graph paper with legs of specified lengths and measure the lengths of their hypotenuses. They record their results in a chart and look for patterns.
- Students create a small right triangle in the middle of a ten-by-ten geoboard or on dot paper and then build squares on each side of the triangle. They record the areas of the squares and look for a relationship involving these areas.
- Students use tangram pieces to build squares on each side of the middle-sized triangular tangram piece. They then describe the relationship among the areas of the three squares.

18. **Explore patterns produced by processes of geometric change, relating iteration, approximation, and fractals.**

- Students use the reducing and/or enlarging feature on a copier to explore repeated reductions/enlargements by the same factor (iteration).
- Students learn about the natural shapes that are produced by growth. They investigate how nature produces complex structures in which basic shapes occur over and over. For example, spider webs, honeycombs, and snowflakes grow larger in a systematic way (iteration). Students measure the age of a tree by looking at its rings; this illustrates approximation. Students produce geometric designs that illustrate these principles, as well as fractals, where miniature versions of the entire design are evident within a small portion of the design.

- Students view the slides which accompany the activity book, *Fractals for the Classroom, Vol. 1*, and determine why each picture might have been included in a book about fractals.
- Students make a table showing the perimeter of a Koch snowflake (a type of fractal) and its area at each stage. They discuss the patterns in the table. This is an example where the perimeter increases without bound but the area approaches a limit.



Stage	Perimeter	Area
0	3	1
1	4	4/3
2	16/3	40/27
3		
4		

19. Investigate, explore, and describe geometry in nature and real-world applications, using models, manipulatives, and appropriate technology.

- Some students read and prepare a report and presentation to the class on *String, Straightedge, and Shadow: The Story of Geometry* by Julia Diggins. Starting with a chapter about the presence of geometry in nature, this story traces the history of geometric discoveries from the invention of early measuring instruments.
- Students model decay in a bacterial culture by cutting a sheet of grid paper in half repeatedly and recording the area of each rectangle in a table. They then graph the number of cuts versus the area to see an example of exponential decay.
- Students investigate the golden ratio ($(1+\sqrt{5})/2$) and its application to architecture (such as the Parthenon), designs of everyday objects such as index cards and picture frames, and its occurrence in pinecones, pineapples, and sunflower seeds.
- Students write about why manufacturers make specially designed containers for packaging their products, indicating how the idea of tessellations might be important in the designs.
- Students use a computer program such as *The Geometry PreSupposer* to investigate the relationship between the lengths of the sides and the measures of the angles in isosceles, scalene, and equilateral triangles.
- Groups of students prepare slide shows using slides from *Geometry in Our World* to illustrate the connections between geometry, science, and art.
- Pairs of students build kites of different shapes, explaining to the class why they chose a particular shape. Each student predicts which kite will fly highest, writing the prediction in his/her journal. The class flies all of the kites, recording the heights of each by using a clinometer and similar triangles.
- Students watch the video *Donald in Mathmagic Land*. Although getting a bit dated, this video still thrills most viewers as Donald Duck encounters many animated applications of

geometry. Students then form teams which focus on an aspect of the video and do further research on that application.

- Students read and choose projects to make from the book *Origami, Japanese Paper Folding* by Florence Sakade or some other origami instruction books. The detailed instructions usually given in such books are rich in mathematical language and discussions among the students should provide a setting for the use of much geometric terminology.

References

Diggins, Julia. *String, Straightedge, and Shadow: The Story of Geometry*. New York: Viking Press, 1965.

Engelhardt, J., Ed. *Geometry in Our World.* Reston, VA: National Council of Teachers of Mathematics, 1992.

New Jersey Department of Education. *Mathematics Instructional Guide*. D. Varygiannes, Coord. Trenton: 1996.

Peitgen, Heinz-Otto, et al. *Fractals for the Classroom: Strategic Activities, Vol. 1*. Reston, VA: National Council of Teachers of Mathematics, and New York: Springer-Verlag, 1991.

Sakade, Florence. *Origami, Japanese Paper Folding*. Rutland, VT: Charles E. Tuttle Co., 1957.

Software

Geometric Golfer. Minnesota Educational Computing Consortium (MECC).

The Geometry PreSupposer. Sunburst Communications.

Tesselmania! Minnesota Educational Computing Consortium (MECC).

Video

Donald in Mathmagic Land. Walt Disney Studios. Los Angeles: The Walt Disney Company, 1959.

General references

Geddes, D. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Geometry in the Middle Grades*. Reston, VA: National Council of Teachers of Mathematics, 1992.

Owens, D. T., Ed. *Research Ideas for the Classroom: Middle Grades Mathematics*. New York: MacMillan, 1993.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 7 — Geometry and Spatial Sense — Grades 9-12

Overview

Students can develop a strong spatial sense from consistent experiences in classroom activities which use a wide variety of manipulatives and technology. The key components of this spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

Geometry has historically held an important role in high school mathematics, primarily through its focus on deductive reasoning and proof; developing skills in deductive reasoning, learning how to construct proofs, and understanding geometric properties are important outcomes of the high school geometry course. Equally important, however, is the continued development of visualization skills, pictorial representations, and applications of geometric ideas since geometry helps students represent and describe the world in which they live and answer questions about natural, physical, and social phenomena.

Deductive reasoning is highly dependent upon understanding and communication skills. In fact, mathematics can be considered as a language — a language of patterns. This language of mathematics must be meaningful if students are to discuss mathematics, construct arguments, and apply geometry productively. Communication and language play a critical role in helping students to construct links between their informal, intuitive geometric notions and the more abstract language and symbolism of high school geometry.

Geometry describes the real world from several viewpoints. One viewpoint is that of standard Euclidean geometry — a deductive system developed from basic axioms. Other widely used viewpoints are those of coordinate geometry, transformational geometry, and vector geometry. The interplay between geometry and algebra strengthens the students' ability to formulate and analyze problems from situations both within and outside mathematics. Although students will at times work separately in synthetic, coordinate, transformational, and vector geometry, they should also have many opportunities to compare, contrast, and translate among these systems. Further, students should learn that certain types of problems are often solved more easily in one specific system than another specific system.

Visualization and pictorial representation are also important aspects of a high school geometry course. Students should have opportunities to visualize and work with two- and three-dimensional figures in order to develop spatial skills fundamental to everyday life and to many careers. By using physical models and other real-world objects, students can develop a strong base for geometric intuition. They can then draw upon these experiences and intuitions when working with abstract ideas.

The goal of high school geometry includes applying geometric ideas to problems in a variety of areas. Each student must develop the ability to solve problems if he or she is to become a productive citizen. Instruction thus must begin with problem situations — not only exercises to be accomplished independently but also problems to be solved in small groups or by the entire class working cooperatively.

Applications of mathematics have changed dramatically over the last twenty years, primarily due to rapid advances in technology. Geometry has, in fact, become more important to students because of computer graphics. Thus, calculators and computers are appropriate and necessary tools in learning geometry.

Students in high school continue to develop their understanding of **spatial relationships.** They construct models from two-dimensional representations of objects, they interpret two- and three-dimensional representations of geometric objects, and they construct two-dimensional representations of actual objects.

Students formalize their understanding of **properties of geometric figures**, using known properties to deduce new relationships. Specific figures which are studied include polygons, circles, prisms, cylinders, pyramids, cones, and spheres. Properties considered should include congruence, similarity, symmetry, measures of angles (especially special relationships such as supplementary and complementary angles), parallelism, and perpendicularity.

In high school, students apply the principles of **geometric transformations** and **coordinate geometry** that they learned in the earlier grades, using these to help develop further understanding of geometric concepts and to establish justifications for conclusions inferred about geometric objects and their relationships. They also begin to use vectors to represent geometric situations.

The **geometry of measurement** is extended in the high school grades to include formalizing procedures for finding perimeters, circumferences, areas, volumes, and surface areas, and solving indirect measurement problems using trigonometric ratios. Students should also use trigonometric functions to model periodic phenomena, establishing an important connection between geometry and algebra.

Students use a variety of geometric representations in **geometric modeling** at these grade levels, such as graphs of algebraic functions on coordinate grids, networks composed of vertices and edges, vectors, transformations, and right triangles to solve problems involving trigonometry. They also explore and analyze further the patterns produced by geometric change.

Deductive reasoning takes on an increasingly important role in the high school years. Students use inductive reasoning as they look for patterns and make conjectures; they use deductive reasoning to justify their conjectures and present reasonable explanations.

Standard 7 — Geometry and Spatial Sense — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11 and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

16[*]. Develop, understand, and apply a variety of strategies for determining perimeter, area, surface area, angle measure, and volume.

- Students find volumes of objects formed by combining geometric figures and develop formulas describing what they have done. For example, they might generate a formula for finding the volume of a silo composed of a cylinder of specified radius and height topped by a hemisphere of the same radius.
- Students construct models to show how the volume of a pyramid with a square base and height equal to a side of the base is related to the volume of a cube with the same base.
- Students develop and use a spreadsheet to determine what the dimensions should be for a cylinder with a fixed volume, in order to minimize the surface area. Similarly, they investigate what should be the dimensions for a rectangle having a fixed perimeter in order to maximize the enclosed area. They discuss how the symmetry of these figures relates to the solutions.

19[*]. Investigate, explore, and describe geometry in nature and real-world applications, using models, manipulatives, and appropriate technology.

- Students use a computer-aided design (CAD) program to investigate rotations of objects in three dimensions.
- Students use *The Geometric SuperSupposer* to measure components of shapes and make observations. For example, they might construct parallelograms and measure sides, angles, and diagonals, observing that opposite sides are congruent, as are opposite angles, and that diagonals bisect each other.
- Students use *The Geometer's Sketchpad* to investigate the effects of rotating a triangle about a fixed point.
- Students use commercial materials such as GeoShapes or Polydrons to construct three-dimensional geometric figures. They make tables concerning the number of vertices, edges, and faces in each solid. They measure the figures to determine their surface areas and volumes. They lay the patterns out flat to examine the nets of each solid. [A net is a flat shape which when folded along indicated lines will produce a three-dimensional object; for

[*] Activities are included here for Indicators 16 and 19, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

example, six identical squares joined in the shape of a cross can be folded to form a cube. Tabs added to the net facilitate attaching appropriate edges so that the shape remains three-dimensional.]

- Students work through the *Ice Cones* lesson that is described in the First Four Standards of this *Framework*. Students create a variety of paper cones out of circles with radius 10 inches which are cut along a radius. They use graphing calculators to find the maximum volume of such cones.
- Students copy geometric designs using compass and straightedge, and generate their own designs.
- Students investigate wallpaper patterns, classifying them according to the transformations used. They study the structure of crystals from a geometric perspective.

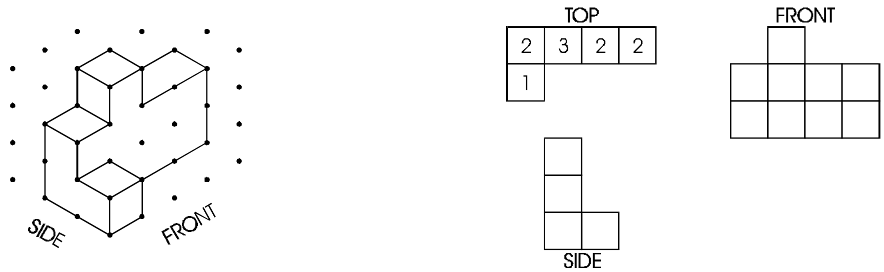
20. Understand and apply properties involving angles, parallel lines, and perpendicular lines.

- Students make tessellations with an assortment of different triangles, noting the variety of geometric patterns that are formed, including parallel lines, congruent angles, congruent triangles, similar triangles, parallelograms, and trapezoids.
- Students identify congruent angles on a parallelogram grid, and use their results to develop conjectures about alternate interior angles, corresponding angles of parallel lines, and opposite angles of a parallelogram.
- Working together, students review geometric vocabulary by sorting words written on index cards into groups and explaining their reasons for creating the groups they did. For example, they might place "parallelogram," "rhombus," "square," and "rectangle" in one group (since they are all parallelograms) and place "kite" and "trapezoid" in another group (since they are not parallelograms).
- Students find a variety of strategies for demonstrating that the sum of the measures of the angles of a triangle is $180°$. Some use protractors and measure a pencil-and-paper figure, others create a triangle with *Geometer's Sketchpad* software and post the measures of the angles before dragging it from a vertex to notice that the sum always remains the same, and still others use a method that requires tearing each of the corners from an oaktag triangle and then fitting them together to make a line.

21. Analyze properties of three-dimensional shapes by constructing models and by drawing and interpreting two-dimensional representations of them.

- Pairs of students work together to describe and draw geometric figures. One student is given a picture involving one or more geometric figures and must describe the drawing to the other student without using the names of the figures. The second student, without seeing the figure, must visualize and represent the picture.
- Students create wind-up posterboard models of rotational three-dimensional solids. They cut out a plane figure such as a circle or a rectangle from posterboard, punch two holes in it near its edges, thread a cut rubberband through the holes, and attach the ends of the rubberband to the ends of a coathanger from which the horizontal wire has been removed. They then twist the rubber band to wind up the figure and release to "show" the solid.
- Students use isometric dot paper to sketch figures made up of cubes. They also sketch top,

front, and side views (projections) of the figure.



- One long-term project that some high school teachers use for assessment is the following: Using a variety of means and materials, students begin by constructing models of the Platonic solids and other three dimensional geometric figures. They are then challenged to work in teams to find a relationship among the number of faces, vertices, and edges that holds for all of the solids (Euler's Formula: F + V E = 2).
- Students read *Flatland: A Romance of Many Dimensions* by Edwin Abbott, a fascinating and imaginative story about life in a two-dimensional world.
- Students use a computer-aided design (CAD) program to investigate rotations of objects in three dimensions.

22. Use transformations, coordinates, and vectors to solve problems in Euclidean geometry.

- Students construct a polygon that outlines the top view of their school. They are asked to imagine that they are architects who need to send this outline by computer to a builder who has no graphics imaging capabilities. They develop strategies for sending this information to the builder. One group locates one corner of the building at the origin and determines the coordinates for the other vertices. Another group uses vectors to tell the builder what direction to proceed from the initial corner located at the origin.
- Students work on the question of where a power transformer should be located on a line so that the length of the cable needed to run to two points not on that line is minimized. They find that if the two points are on the same side of the line, then by using reflections they can construct a straight line that crosses the given line at the desired location.
- Students first determine the coordinates for the vertices of a parallelogram, a rhombus, a rectangle, an isosceles trapezoid, and a square with one vertex at the origin and a side along the x-axis. They then work in groups to determine where the coordinate system should be placed to simplify the coordinate selection for a kite, a rhombus, and a square.
- Students draw two congruent triangles anywhere in the plane and determine the minimum number of reflections needed to map one onto the other.
- Students draw a triangle on graph paper and then find the image of the triangle when the coordinates of each vertex are multiplied by various constants. They draw each resulting triangle and determine its area. They make a table of their results and look for relationships between the constants used for dilation and the ratios of the areas.
- Students use a Mira (Reflecta) to find the center of a circle, to draw the perpendicular

bisectors of a line segment, or to draw the medians of a triangle.

- Students apply transformations to figures drawn on coordinate grids, record the coordinates of the original figure and its image, and look for patterns. They express these patterns verbally and symbolically. For example, flipping a point across the x-axis changes the sign of the y-coordinate so that the point (x,y) moves to (x, -y).
- Given the equation of a line, students plot the line on a coordinate grid and then shift the line according to a given translation. They then determine the equation of the resulting line. After doing several such problems, students identify patterns that they have found and write conjectures.
- Students work through the *Building Parabolas* lesson that is described in the First Four Standards of this *Framework*. They investigate the effects of various coefficients on the general shape of a parabola and connect these to geometric transformations.

23. Use basic trigonometric ratios to solve problems involving indirect measurement.

- Students use trigonometric ratios to determine distances which cannot be measured directly, such as the distance between two points on opposite sides of a chasm.
- Students investigate how the paths of tunnels are determined so that people digging from each end wind up in the same place.
- Students use trigonometry to determine the cloud ceiling at night by directing a light (kept in a narrow beam by a parabolic reflector) toward the clouds. An observer at a specified distance measures the angle of elevation to the point at which the light is reflected from the cloud.
- Students plot the average high temperature for each month over the course of five years to see an example of a periodic function. They discuss what types of functions might be appropriate to represent this relationship.

24. Solve real-world and mathematical problems using geometric models.

- Students visit a construction site where the “framing” step of a building process is taking place. They note where congruence occurs (such as in the beams of the roof, where angles must be congruent). They write about why congruence is essential to buildings and other structures.
- Students use paper fasteners and tagboard strips with a hole punched near each end to investigate the rigidity of various polygon shapes. For shapes that are not rigid, they determine how to make the shape more rigid.
- Students draw a geometric representation and develop a formula to solve the problem of how many handshakes will take place if there are *n* people and each person shakes hands with each other person exactly once.
- Students work through the *On the Boardwalk* lesson that is described in the Introduction to this *Framework*. They determined the probability of winning a prize when tossing a coin onto a grid by having the coin avoid all of the grid lines.
- Students use graph models to represent a situation in which a large company wishes to install a pneumatic tube system that would enable small items to be sent between any of ten

locales, possibly by relay. Given the cost associated with possible tubes (edges), the students work in groups to determine optimal pneumatic tube systems for the company. They report their results in letters written individually to the company president.

- Students work through the *Making Rectangles* lesson that is described in the First Four Standards of this *Framework*. They use combinations of algebra tiles which they try to arrange into rectangle shapes to help them develop procedures for multiplying binomials and factoring polynomials.

25. Use inductive and deductive reasoning to solve problems and to present reasonable explanations of and justifications for the solutions.

- In a computer-based, open-ended, assessment, groups of students use computer software to draw parallelograms, make measurements, and list as many properties of parallelograms and their diagonals as they can.
- Students prove deductively that a parallelogram is divided into congruent triangle by a diagonal. They also prove that any angle inscribed in a semi-circle is a right angle. (An angle ABC is inscribed in a semi-circle if AC is a diameter and B is any other point on the circle.)
- Students explain in writing to a friend what the formula is for the measure of each interior angle in a regular polygon with *n* sides and how it is derived.
- Students build staircases from cubes, recording the number of steps and the total number of cubes used for each construction. They look for patterns, expressing them in words and symbolically in equations. They then try to justify their results using deductive reasoning.
- Students use *Cabri* software to investigate what happens when consecutive midpoints of a quadrilateral are connected in order. They state a conjecture based on their investigation and explain why they think it is true.
- Students investigate the relationship between the number of diagonals that can be drawn from one vertex of a polygon and the number of sides of that polygon. They write about their findings in their journals.
- Students work through the *A Sure Thing!?* lesson in the Introduction to this *Framewor*k. They investigate the number of non-overlapping regions that can be created if they draw all the chords joining *n* points on the circumference of a circle.

26. Analyze patterns produced by processes of geometric change and express them in terms of iteration, approximation, limits, self-similarity, and fractals.

- Students duplicate the beginning stages of a fractal construction in the plane and analyze the sequences of their perimeters and their areas.
- Students use the reduction and enlargement capabilities of a copy machine to investigate the effects on area. They make a table showing the linear rate of reduction/enlargement and the resulting area for each successive reduction/enlargement. Then they graph the results — an exponential function showing either decay or growth.
- Students use the slides and appropriate activities from *Fractals for the Classroom, Vol. 1* to analyze patterns produced by changes in geometric shapes.

- Students model decay in a bacterial culture by cutting a sheet of grid paper in half repeatedly and recording the area of each rectangle in a table. They then graph the number of cuts versus the area to see an example of exponential decay.
- Students plot the relationship between body height and arm length for people from one year of age through adulthood on coordinate grid paper and on log-log paper. They see that the graph is not a straight line on the coordinate grid paper; it is actually a logarithmic function. They find that the function appears as a straight line on log-log paper.

27. Explore applications of other geometries in real-world contexts.

- Students represent lines using string and pins on styrofoam balls (spheres). They analyze the properties of lines (e.g., all lines intersect) and triangles (e.g., it is possible to have a triangle with three 90° angles). They apply their results to finding the shortest route between two points on the earth.
- Students investigate the angel and devil drawings of M. C. Escher as examples of geometries in which there may be many "lines" through a given point that do not intersect a given "line." In this case, a "line" is an arc of a circle that is perpendicular to the outside circle of the drawing.
- Students explore another geometry using *Non-Euclidean Adventures on the Lénárt Sphere*.
- Students determine how many people are needed on a committee if there are to be four subcommittees, with each person on two subcommittees and each pair of subcommittees having one person in common. Most groups use letters to represent the individuals, and represent the four subcommittees by collections of letters, as in the following proposed solution {ABC, ADE, BDF, CEF}. The teacher asks the students to make a diagram of their solution, using "points" for people and "lines" for subcommittees, so that each subcommittee is a line whose points are its members. The rules for subcommittees become axioms about these points and lines; for example, "each person is on two subcommittees" becomes "each point is on two lines." The resulting geometry is an example of a finite geometry.

References

Abbott, Edwin. *Flatland: A Romance of Many Dimensions*. New York: Viking, 1987.

Lénárt, István. *Non-Euclidean Adventures on the Lénárt Sphere*. Berkeley, CA: Key Curriculum Press, 1996.

Peitgen, Heinz-Otto, et al. *Fractals for the Classroom: Strategic Activities, Vol. 1*. Reston, VA: National Council of Teachers of Mathematics, and New York: Springer-Verlag, 1991.

Software

Cabri. Texas Instruments.

Geometer's Sketchpad. Key Curriculum Press.

Geometric Golfer. Minnesota Educational Computing Consortium (MECC).

The Geometry PreSupposer. Sunburst Communications.

Tesselmania! Minnesota Educational Computing Consortium (MECC).

General reference

Coxford, A.F. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Geometry from Multiple Perspectives*. Reston, VA: National Council of Teachers of Mathematics, 1991.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.