

NEW JERSEY

STUDENT LEARNING STANDARDS FOR

Mathematics | **Grade 8**

## Mathematics | Grade 8

In Grade 8, instructional time should focus on three critical areas: (1) formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation, and solving linear equations and systems of linear equations; (2) grasping the concept of a function and using functions to describe quantitative relationships; (3) analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem.

(1) Students use linear equations and systems of linear equations to represent, analyze, and solve a variety of problems. Students recognize equations for proportions ( $y/x = m$  or  $y = mx$ ) as special linear equations ( $y = mx + b$ ), understanding that the constant of proportionality ( $m$ ) is the slope, and the graphs are lines through the origin. They understand that the slope ( $m$ ) of a line is a constant rate of change, so that if the input or  $x$ -coordinate changes by an amount  $A$ , the output or  $y$ -coordinate changes by the amount  $m \cdot A$ . Students also use a linear equation to describe the association between two quantities in bivariate data (such as arm span vs. height for students in a classroom). At this grade, fitting the model, and assessing its fit to the data are done informally. Interpreting the model in the context of the data requires students to express a relationship between the two quantities in question and to interpret components of the relationship (such as slope and  $y$ -intercept) in terms of the situation.

Students strategically choose and efficiently implement procedures to solve linear equations in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equation. Students solve systems of two linear equations in two variables and relate the systems to pairs of lines in the plane; these intersect, are parallel, or are the same line. Students use linear equations, systems of linear equations, linear functions, and their understanding of slope of a line to analyze situations and solve problems.

(2) Students grasp the concept of a function as a rule that assigns to each input exactly one output. They understand that functions describe situations where one quantity determines another. They can translate among representations and partial representations of functions (noting that tabular and graphical representations may be partial representations), and they describe how aspects of the function are reflected in the different representations.

(3) Students use ideas about distance and angles, how they behave under translations, rotations, reflections, and dilations, and ideas about congruence and similarity to describe and analyze two-dimensional figures and to solve problems. Students show that the sum of the angles in a triangle is the angle formed by a straight line, and that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds, for

example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons. Students complete their work on volume by solving problems involving cones, cylinders, and spheres.

## Grade 8 Overview

### The Number System

- Know that there are numbers that are not rational, and approximate them by rational numbers.

### Expressions and Equations

- Work with radicals and integer exponents.
- Understand the connections between proportional relationships, lines, and linear equations.
- Analyze and solve linear equations and pairs of simultaneous linear equations.

### Functions

- Define, evaluate, and compare functions.
- Use functions to model relationships between quantities.

### Geometry

- Understand congruence and similarity using physical models, transparencies, or geometry software.
- Understand and apply the Pythagorean Theorem.
- Solve real-world and mathematical problems involving volume of cylinders, cones and spheres.

### Statistics and Probability

- Investigate patterns of association in bivariate data.

### Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning

## The Number System

8.NS

**A. Know that there are numbers that are not rational, and approximate them by rational numbers.**

1. Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.
2. Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g.,  $\pi^2$ ). *For example, by truncating the decimal expansion of  $\sqrt{2}$ , show that  $\sqrt{2}$  is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*

## Expressions and Equations

8.EE

**A. Work with radicals and integer exponents.**

1. Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example,  $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$ .*
2. Use square root and cube root symbols to represent solutions to equations of the form  $x^2 = p$  and  $x^3 = p$ , where  $p$  is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that  $\sqrt{2}$  is irrational.
3. Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. *For example, estimate the population of the United States as  $3 \times 10^8$  and the population of the world as  $7 \times 10^9$ , and determine that the world population is more than 20 times larger.*
4. Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

**B. Understand the connections between proportional relationships, lines, and linear equations.**

5. Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. *For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.*
6. Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y = mx$  for a line through the origin and the equation  $y = mx + b$  for a line intercepting the vertical axis at  $b$ .

**C. Analyze and solve linear equations and pairs of simultaneous linear equations.**

7. Solve linear equations in one variable.
  - a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form  $x = a$ ,  $a = a$ , or  $a = b$  results (where  $a$  and  $b$  are different numbers).

- b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.
- 8. Analyze and solve pairs of simultaneous linear equations.
  - a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
  - b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example,  $3x + 2y = 5$  and  $3x + 2y = 6$  have no solution because  $3x + 2y$  cannot simultaneously be 5 and 6.*
  - c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

## Functions

8.F

### A. Define, evaluate, and compare functions.

1. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.<sup>1</sup>
2. Compare properties (e.g. rate of change, intercepts, domain and range) of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*
3. Interpret the equation  $y = mx + b$  as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function  $A = s^2$  giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.*

### B. Use functions to model relationships between quantities.

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
5. Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

## Geometry

8.G

### A. Understand congruence and similarity using physical models, transparencies, or geometry software.

1. Verify experimentally the properties of rotations, reflections, and translations:
  - a. Lines are transformed to lines, and line segments to line segments of the same length.
  - b. Angles are transformed to angles of the same measure.
  - c. Parallel lines are transformed to parallel lines.

<sup>1</sup>Function notation is not required in Grade 8.

2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
3. Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
4. Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.
5. Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.*

## **B. Understand and apply the Pythagorean Theorem.**

6. Explain a proof of the Pythagorean Theorem and its converse.
7. Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

## **C. Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.**

9. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

## **Statistics and Probability**

**8.SP**

### **A. Investigate patterns of association in bivariate data.**

1. Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.
2. Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit (e.g. line of best fit) by judging the closeness of the data points to the line.
3. Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. *For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.*
4. Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. *For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?*

## Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

### **1 Make sense of problems and persevere in solving them.**

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

### **2 Reason abstractly and quantitatively.**

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

### **3 Construct viable arguments and critique the reasoning of others.**

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of

others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

#### **4 Model with mathematics.**

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

#### **5 Use appropriate tools strategically.**

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

#### **6 Attend to precision.**

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students

give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

## **7 Look for and make use of structure.**

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see  $7 \times 8$  equals the well remembered  $7 \times 5 + 7 \times 3$ , in preparation for learning about the distributive property. In the expression  $x^2 + 9x + 14$ , older students can see the 14 as  $2 \times 7$  and the 9 as  $2 + 7$ . They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see  $5 - 3(x - y)^2$  as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers  $x$  and  $y$ .

## **8 Look for and express regularity in repeated reasoning.**

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation  $(y - 2)/(x - 1) = 3$ . Noticing the regularity in the way terms cancel when expanding  $(x - 1)(x + 1)$ ,  $(x - 1)(x^2 + x + 1)$ , and  $(x - 1)(x^3 + x^2 + x + 1)$  might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

## Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards, which set an expectation of understanding, are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

## Glossary

**Addition and subtraction within 5, 10, 20, 100, or 1000.** Addition or subtraction of two whole numbers with whole number answers, and with sum or minuend in the range 0-5, 0-10, 0-20, or 0-100, respectively. Example:  $8 + 2 = 10$  is an addition within 10,  $14 - 5 = 9$  is a subtraction within 20, and  $55 - 18 = 37$  is a subtraction within 100.

**Additive inverses.** Two numbers whose sum is 0 are additive inverses of one another. Example:  $3/4$  and  $-3/4$  are additive inverses of one another because  $3/4 + (-3/4) = (-3/4) + 3/4 = 0$ .

**Associative property of addition.** See Table 3 in this Glossary.

**Associative property of multiplication.** See Table 3 in this Glossary.

**Bivariate data.** Pairs of linked numerical observations. Example: a list of heights and weights for each player on a football team.

**Box plot.** A method of visually displaying a distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data.<sup>1</sup>

**Commutative property.** See Table 3 in this Glossary.

**Complex fraction.** A fraction  $A/B$  where  $A$  and/or  $B$  are fractions ( $B$  nonzero).

**Computation algorithm.** A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. *See also:* computation strategy.

**Computation strategy.** Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. *See also:* computation algorithm.

**Congruent.** Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations).

**Counting on.** A strategy for finding the number of objects in a group without having to count every member of the group. For example, if a stack of books is known to have 8 books and 3 more books are added to the top, it is not necessary to count the stack all over again. One can find the total by *counting on*—pointing to the top book and saying “eight,” following this with “nine, ten, eleven. There are eleven books now.”

**Dot plot.** *See:* line plot.

**Dilation.** A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

**Expanded form.** A multi-digit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. For example,  $643 = 600 + 40 + 3$ .

**Expected value.** For a random variable, the weighted average of its possible values, with weights given by their respective probabilities.

**First quartile.** For a data set with median  $M$ , the first quartile is the median of the data values less than  $M$ . Example: For the data set  $\{1, 3, 6, 7, 10, 12, 14, 15, 22, 120\}$ , the first quartile is 6.<sup>2</sup> *See also:* median, third quartile, interquartile range.

**Fraction.** A number expressible in the form  $a/b$  where  $a$  is a whole number and  $b$  is a positive whole number. (The word *fraction* in these standards always refers to a non-negative number.) *See also:* rational number.

**Identity property of 0.** See Table 3 in this Glossary.

**Independently combined probability models.** Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

<sup>1</sup>Adapted from Wisconsin Department of Public Instruction, <http://dpi.wi.gov/standards/mathglos.html>, accessed March 2, 2010.

<sup>2</sup>Many different methods for computing quartiles are in use. The method defined here is sometimes called the Moore and McCabe method. See Langford, E., “Quartiles in Elementary Statistics,” *Journal of Statistics Education* Volume 14, Number 3 (2006).

**Integer.** A number expressible in the form  $a$  or  $-a$  for some whole number  $a$ .

**Interquartile Range.** A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the interquartile range is  $15 - 6 = 9$ . *See also:* first quartile, third quartile.

**Line plot.** A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a dot plot.<sup>3</sup>

**Mean.** A measure of center in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list.<sup>4</sup> Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean is 21.

**Mean absolute deviation.** A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean absolute deviation is 20.

**Median.** A measure of center in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list—or the mean of the two central values, if the list contains an even number of values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 90}, the median is 11.

**Midline.** In the graph of a trigonometric function, the horizontal line halfway between its maximum and minimum values.

**Multiplication and division within 100.** Multiplication or division of two whole numbers with whole number answers, and with product or dividend in the range 0-100. Example:  $72 \div 8 = 9$ .

**Multiplicative inverses.** Two numbers whose product is 1 are multiplicative inverses of one another. Example:  $3/4$  and  $4/3$  are multiplicative inverses of one another because  $3/4 \times 4/3 = 4/3 \times 3/4 = 1$ .

**Number line diagram.** A diagram of the number line used to represent numbers and support reasoning about them. In a number line diagram for measurement quantities, the interval from 0 to 1 on the diagram represents the unit of measure for the quantity.

**Percent rate of change.** A rate of change expressed as a percent. Example: if a population grows from 50 to 55 in a year, it grows by  $5/50 = 10\%$  per year.

**Probability distribution.** The set of possible values of a random variable with a probability assigned to each.

**Properties of operations.** See Table 3 in this Glossary.

**Properties of equality.** See Table 4 in this Glossary.

**Properties of inequality.** See Table 5 in this Glossary.

**Properties of operations.** See Table 3 in this Glossary.

**Probability.** A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, or testing for a medical condition).

**Probability model.** A probability model is used to assign probabilities to outcomes of a chance process by examining the nature of the process. The set of all outcomes is called the sample space, and their probabilities sum to 1. *See also:* uniform probability model.

**Random variable.** An assignment of a numerical value to each outcome in a sample space.

**Rational expression.** A quotient of two polynomials with a non-zero denominator.

**Rational number.** A number expressible in the form  $a/b$  or  $-a/b$  for some fraction  $a/b$ . The rational numbers include the integers.

**Rectilinear figure.** A polygon all angles of which are right angles.

**Rigid motion.** A transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

<sup>3</sup>Adapted from Wisconsin Department of Public Instruction, *op. cit.*

<sup>4</sup>To be more precise, this defines the *arithmetic mean*.

**Repeating decimal.** The decimal form of a rational number. *See also:* terminating decimal.

**Sample space.** In a probability model for a random process, a list of the individual outcomes that are to be considered.

**Scatter plot.** A graph in the coordinate plane representing a set of bivariate data. For example, the heights and weights of a group of people could be displayed on a scatter plot.<sup>5</sup>

**Similarity transformation.** A rigid motion followed by a dilation

**Tape diagram.** A drawing that looks like a segment of tape, used to illustrate number relationships. Also known as a strip diagram, bar model, fraction strip, or length model.

**Terminating decimal.** A decimal is called terminating if its repeating digit is 0.

**Third quartile.** For a data set with median  $M$ , the third quartile is the median of the data values greater than  $M$ . Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the third quartile is 15. *See also:* median, first quartile, interquartile range.

**Transitivity principle for indirect measurement.** If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

**Uniform probability model.** A probability model which assigns equal probability to all outcomes. *See also:* probability model.

**Vector.** A quantity with magnitude and direction in the plane or in space, defined by an ordered pair or triple of real numbers.

**Visual fraction model.** A tape diagram, number line diagram, or area model.

**Whole numbers.** The numbers 0, 1, 2, 3, ....

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<sup>5</sup>Adapted from Wisconsin Department of Public Instruction, *op. cit.*

Table 1. Common addition and subtraction situations.<sup>6</sup>

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now?  $2 + 3 = ?$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two?  $2 + ? = 5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before?  $? + 3 = 5$
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now?  $5 - 2 = ?$	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat?  $5 - ? = 3$	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before?  $? - 2 = 3$
	Total Unknown	Addend Unknown	Both Addends Unknown <sup>1</sup>
Put Together/ Take Apart <sup>2</sup>	Three red apples and two green apples are on the table. How many apples are on the table?  $3 + 2 = ?$	Five apples are on the table. Three are red and the rest are green. How many apples are green?  $3 + ? = 5, 5 - 3 = ?$	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase?  $5 = 0 + 5, 5 = 5 + 0$ $5 = 1 + 4, 5 = 4 + 1$ $5 = 2 + 3, 5 = 3 + 2$
	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare <sup>3</sup>	(“How many more?” version):  Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy?  (“How many fewer?” version):  Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie?  $2 + ? = 5, 5 - 2 = ?$	(Version with “more”):  Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have?  (Version with “fewer”):  Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have?  $2 + 3 = ?, 3 + 2 = ?$	(Version with “more”):  Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have?  (Version with “fewer”):  Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have?  $5 - 3 = ?, ? + 3 = 5$

<sup>1</sup>These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as.

<sup>2</sup>Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

<sup>3</sup>For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using more for the bigger unknown and using less for the smaller unknown). The other versions are more difficult.

<sup>6</sup>Adapted from Box 2-4 of National Research Council (2009, op. cit., pp. 32, 33).

Table 2. Common multiplication and division situations.<sup>7</sup>

	Unknown Product	Group Size Unknown ("How many in each group?" Division)	Number of Groups Unknown ("How many groups?" Division)
	$3 \times 6 = ?$	$3 \times ? = 18$ , and $18 \div 3 = ?$	$? \times 6 = 18$ , and $18 \div 6 = ?$
Equal Groups	<p>There are 3 bags with 6 plums in each bag. How many plums are there in all?</p> <p><i>Measurement example.</i> You need 3 lengths of string, each 6 inches long. How much string will you need altogether?</p>	<p>If 18 plums are shared equally into 3 bags, then how many plums will be in each bag?</p> <p><i>Measurement example.</i> You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?</p>	<p>If 18 plums are to be packed 6 to a bag, then how many bags are needed?</p> <p><i>Measurement example.</i> You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?</p>
Arrays, <sup>4</sup> Area <sup>5</sup>	<p>There are 3 rows of apples with 6 apples in each row. How many apples are there?</p> <p><i>Area example.</i> What is the area of a 3 cm by 6 cm rectangle?</p>	<p>If 18 apples are arranged into 3 equal rows, how many apples will be in each row?</p> <p><i>Area example.</i> A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?</p>	<p>If 18 apples are arranged into equal rows of 6 apples, how many rows will there be?</p> <p><i>Area example.</i> A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?</p>
Compare	<p>A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost?</p> <p><i>Measurement example.</i> A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?</p>	<p>A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost?</p> <p><i>Measurement example.</i> A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?</p>	<p>A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat?</p> <p><i>Measurement example.</i> A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?</p>
General	$a \times b = ?$	$a \times ? = p$ , and $p \div a = ?$	$? \times b = p$ , and $p \div b = ?$

<sup>4</sup>The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

<sup>5</sup>Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

<sup>7</sup>The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

**Table 3.** The properties of operations. Here  $a$ ,  $b$  and  $c$  stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

<i>Associative property of addition</i>	$(a + b) + c = a + (b + c)$
<i>Commutative property of addition</i>	$a + b = b + a$
<i>Additive identity property of 0</i>	$a + 0 = 0 + a = a$
<i>Existence of additive inverses</i>	For every $a$ there exists $-a$ so that $a + (-a) = (-a) + a = 0$ .
<i>Associative property of multiplication</i>	$(a \times b) \times c = a \times (b \times c)$
<i>Commutative property of multiplication</i>	$a \times b = b \times a$
<i>Multiplicative identity property of 1</i>	$a \times 1 = 1 \times a = a$
<i>Existence of multiplicative inverses</i>	For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$ .
<i>Distributive property of multiplication over addition</i>	$a \times (b + c) = a \times b + a \times c$

**Table 4.** The properties of equality. Here  $a$ ,  $b$  and  $c$  stand for arbitrary numbers in the rational, real, or complex number systems.

<i>Reflexive property of equality</i>	$a = a$
<i>Symmetric property of equality</i>	If $a = b$ , then $b = a$ .
<i>Transitive property of equality</i>	If $a = b$ and $b = c$ , then $a = c$ .
<i>Addition property of equality</i>	If $a = b$ , then $a + c = b + c$ .
<i>Subtraction property of equality</i>	If $a = b$ , then $a - c = b - c$ .
<i>Multiplication property of equality</i>	If $a = b$ , then $a \times c = b \times c$ .
<i>Division property of equality</i>	If $a = b$ and $c \neq 0$ , then $a \div c = b \div c$ .
<i>Substitution property of equality</i>	If $a = b$ , then $b$ may be substituted for $a$ in any expression containing $a$ .

**Table 5.** The properties of inequality. Here  $a$ ,  $b$  and  $c$  stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$ , $a = b$ , $a > b$ .
If $a > b$ and $b > c$ then $a > c$ .
If $a > b$ , then $b < a$ .
If $a > b$ , then $-a < -b$ .
If $a > b$ , then $a \pm c > b \pm c$ .
If $a > b$ and $c > 0$ , then $a \times c > b \times c$ .
If $a > b$ and $c < 0$ , then $a \times c < b \times c$ .
If $a > b$ and $c > 0$ , then $a \div c > b \div c$ .
If $a > b$ and $c < 0$ , then $a \div c < b \div c$ .